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An Inner Approximation Method for Optimization over the Weakly Efficient Set

SYUUJI YAMADA, TETSUZO TANINO and MASAHIRO INUIGUCHI
Department of Electronics and Information Systems, Graduate School of Engineering, Osaka University, Osaka 565-0871, Japan (e-mail: yamada@eie.eng.osaka-u.ac.jp

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Abstract. In this paper, we consider an optimization problem which aims to minimize a convex function over the weakly efficient set of a multiobjective programming problem. To solve such a problem, we propose an inner approximation algorithm, in which two kinds of convex subproblems are solved successively. These convex subproblems are fairly easy to solve and therefore the proposed algorithm is practically useful. The algorithm always terminates after finitely many iterations by compromising the weak efficiency to a multiobjective programming problem. Moreover, for a subproblem which is solved at each iteration of the algorithm, we suggest a procedure for eliminating redundant constraints.

Key words: Weakly Efficient Set, Global Optimization, Dual Problem, Inner Approximation Method

1. Introduction

We consider the following multiobjective programming problem:

$$(MOP)\begin{cases} \text{maximize} & \langle c^i, x\rangle, \qquad i = 1, \dots, K, \\ \text{subject to} & x \in X \subset R^n, \end{cases}$$

where X is a compact convex set and $\langle \cdot , \cdot \rangle$ denotes the Euclidean inner product in R^n. The objective functions $\langle c^i, x\rangle$, $i = 1, \dots, K$, express the criteria which the decision-maker wants to maximize. A feasible vector $x \in X$ is said to be weakly efficient if there is no feasible vector y such that $\langle c^i, x\rangle < \langle c^i, y\rangle$ for every $i \in \{1, \dots, K\}$. The set X_e of all feasible weakly efficient vectors is called the weakly efficient set. From the compactness of X, the weakly efficient set X_e is not empty. For problem (MOP), we shall assume the following throughout this paper:

(A1) $X = \{x \in R^n : p_j(x) \leqslant 0,\ j = 1, \dots, t\}$ where $p_j : R^n \to R$, $j = 1, \dots, t$, are differentiable convex functions satisfying $p_j(0) < 0$ (whence $0 \in \operatorname{int} X$),

(A2) $\{x \in R^n : \langle c^i, x\rangle < 0 \text{ for all } i \in \{1, \dots, K\}\} \neq \emptyset$.

Let $p(x) := \max_{j=1,\dots,t} p_j(x)$. Then $X := \{x \in R^n : p(x) \leqslant 0\}$ and int $X = \{x \in R^n : p(x) < 0\} \neq \emptyset$ (Slater's constraint qualification).

In this paper, we consider a convex cost function minimization problem over the weakly efficient set. An example of such a problem is furnished by the portfolio optimization problem in capital markets. A fund manager may look for a portfolio which minimizes the transaction cost on the efficient set. In case X is a polytope, Thach et al. (1996) has proposed a cutting plane method for solving the problem. In order to solve the problem in case X is not necessarily a polytope but a compact convex set, we propose an inner approximation method.

The organization of this paper is as follows. In Section 2, we explain a convex function minimization over the weakly efficient set in R^n. Moreover, we reformulate problem (OES) (minimizing a convex function over the weakly efficient set) as an equivalent problem (MP) of minimizing a quasi-convex function over the complement of a convex set containing 0 in its interior. Following Thach's duality theory (1993, 1994), to this problem (MP) we associate a dual problem (DP), which consists in maximizing a quasi-convex function over a compact convex set. In Section 3, we formulate an inner approximation algorithm for the problem, and establish the convergence of the algorithm. In Section 4, we propose a criterion for finite termination of the algorithm. In Section 5, for the sake of computational efficiency, we propose a procedure for identifying redundant constraints for the subproblem.

Throughout this paper, we use the following notation: int X, bd X and co X denote the interior set of $X \subset R^n$, the boundary set of X and the convex hull of X, respectively. $\bar{R} = R \cup \{-\infty\} \cup \{+\infty\}$. Given a convex polyhedral set (or polytope) $X \subset R^n$, $V(X)$ denotes the set of all vertices of X. For a subset $X \subset R^n$, $X^\circ = \{u \in R^n : \langle u, x\rangle \leqslant 1,\ \forall x \in X\}$ is called the polar set of X. For a subset $X \subset R^n$, the indicator of X : $\delta(\,\cdot\,|X)$ is an extended-real-valued function defined as follows:

$$\delta(x|X) = \begin{cases} 0 & \text{if } x \in X \\ +\infty & \text{if } x \notin X. \end{cases}$$

Given a function $f : R^n \to R \cup \{+\infty\}$, the quasi-conjugate of f is the function f^H defined as follows:

$$f^H(u) = \begin{cases} -\sup\{f(x) : x \in R^n\} & \text{if } u = 0 \\ -\inf\{f(x) : \langle u, x\rangle \geqslant 1\} & \text{if } u \neq 0 \end{cases}$$

(see for example Konno et al. (1997) and Thach (1993, 1994)). The gradient of f at x is denoted by $\nabla f(x)$ and the subdifferential of f at x by $\partial f(x)$.

2. Minimizing a Convex Function over the Weakly Efficient Set

Let us consider the following problem which seeks to minimize a function f over the weakly efficient set of (MOP):

$$(OES)\begin{cases}\text{minimize} & f(x)\\ \text{subject to} & x\in X_e,\end{cases}$$

where $f : R^n \to R$ satisfies the following assumptions:

(B1) f is a convex function,
(B2) $\arg\min\{f(x) : x \in R^n\} = \{0\}$.

Let $C = \{x \in R^n : \langle c^i, x\rangle \leqslant 0, \text{ for all } i \in \{1, \ldots, K\}\}$. Then, by assumption (A2), $\operatorname{int} C \neq \emptyset$. It follows from the following lemma that the weakly efficient set X_e to problem (MOP) is formulated as $X_e = X\backslash\operatorname{int}(X + C)$.

LEMMA 2.1. *Under assumption (A2),* $X_e = X\backslash\operatorname{int}(X + C)$.

Proof. For any $x \in R^n$, $\{x\} + C$ is formulated as

$$\begin{aligned}\{x\}+C &= \{x\} + \{y \in R^n : \langle c^i, y\rangle \leqslant 0 \text{ for all } i \in \{1,\ldots,K\}\}\\ &= \{z \in R^n : z = x + y,\ \langle c^i, y\rangle \leqslant 0 \text{ for all } i \in \{1,\ldots,K\}\}\\ &= \{z \in R^n : \langle c^i, z - x\rangle \leqslant 0 \text{ for all } i \in \{1,\ldots,K\}\}\\ &= \{z \in R^n : \langle c^i, z\rangle \leqslant \langle c^i, x\rangle \text{ for all } i \in \{1,\ldots,K\}\}.\end{aligned}$$

Hence, we have

$$\operatorname{int}(\{x\} + C) = \{z \in R^n : \langle c^i, z\rangle < \langle c^i, x\rangle \text{ for all } i \in \{1,\ldots,K\}\}. \tag{1}$$

From assumption (A2) and the convexity of X and C, the following equation holds:

$$\operatorname{int}(X + C) = \bigcup_{x\in X} \operatorname{int}(\{x\} + C) \tag{2}$$

(Tanaka and Kuroiwa (1993)).

Let $\bar{x} \in X\backslash\operatorname{int}(X + C)$. Then, we have $\bar{x} \in X$ and by (2), $\bar{x} \notin \bigcap_{x\in X} \operatorname{int}(\{x\} + C)$. It follows from (1) that

$$\nexists x \in X \text{ such that } \langle c^i, \bar{x}\rangle < \langle c^i, x\rangle \text{ for all } i \in \{1,\ldots,K\}.$$

Consequently, every point $\bar{x} \in X\backslash\operatorname{int}(X + C)$ is a weakly efficient solution to problem (MOP), that is, $X_e \supset X\backslash\operatorname{int}(X + C)$.

Conversely, let $\bar{x} \in X_e$. Then $\bar{x}$ is contained in X. By the definition of the weakly efficient set,

$$\nexists x \in X \text{ such that } \langle c^i, \bar{x}\rangle < \langle c^i, x\rangle \text{ for all } i \in \{1,\ldots,K\}.$$

Therefore, by (1), it follows that for any $x \in X$, $\bar{x} \notin \text{int}(\{x\} + C)$. From (2), this implies that $\bar{x} \notin \text{int}(X+C)$. Thus $\bar{x} \notin \text{int}(X+C)$ and hence $X_e \subset X \backslash \text{int}(X+C)$. Consequently, $X_e = X \backslash \text{int}(X + C)$. □

From the compactness of X and Lemma 2.1, we know that X_e is compact and that problem (OES) has an optimal solution. Denote by $\inf(OES)$ the optimal value in problem (OES). Since $f(x) < +\infty$ for any $x \in X_e$, we have $\inf(OES) < +\infty$.

By using the indicator of X, problem (OES) can be reformulated as

$$(MP) \begin{cases} \text{minimize} & g(x) \\ \text{subject to} & x \in R^n \backslash \text{int}(X + C) \end{cases}$$

where $g(x) := f(x) + \delta(x|X)$. The dual problem of problem (MP) is formulated as

$$(DP) \begin{cases} \text{maximize} & g^H(u) \\ \text{subject to} & u \in (X + C)^\circ. \end{cases}$$

It follows from assumption (A1) and the definition of C that the origin is contained in the interior set of $X + C$. Hence, by the principle of the duality, $(X + C)^\circ$ is a compact convex set. Furthermore, since g^H is a quasi-convex function (Thach et al. (1996) and Konno et al. (1997), Chapter 2), we note that problem (DP) is a quasi-convex maximization problem over a compact convex set in R^n. Denote by $\inf(MP)$ and $\sup(DP)$ the optimal values of (MP) and (DP), respectively. Since problem (MP) is equivalent to problem (OES), we have $\inf(MP) = \inf(OES) < +\infty$. Moreover, it follows from the duality relation between problems (MP) and (DP) that $\inf(MP) = -\sup(DP)$ (cf., Thach et al. (1996) and Konno et al. (1997), Chapter 4).

3. An Inner Approximation Method for Problem (MP)

3.1. RELAXED PROBLEMS FOR PROBLEMS (MP) AND (DP)

One of the reasons for difficulty in solving problem (MP) is that the feasible set X of problem (MOP) is not a polytope, so that $X + C$ is not a convex polyhedral set. If $X + C$ is a convex polyhedral set, then the feasible set of problem (MP) can be formulated as the union of finite halfspaces. In that case, problem (MP) is fairly easy to solve by minimizing g over every halfspace.

Therefore, in this subsection, we discuss the following problem:

$$(P) \begin{cases} \text{minimize} & g(x), \\ \text{subject to} & x \in R^n \backslash \text{int}(S + C), \end{cases}$$

where S is a polytope such that $S \subset X$ and $0 \in \text{int}(S + C)$. Then, we get $R^n \backslash \text{int}(S+C) \supset R^n \backslash \text{int}(X+C)$. Therefore, problem (P) is a relaxed problem for problem (MP). From the definition of g, we note that problem (P) is equivalent to minimizing $f(x)$ subject to $x \in X\backslash\text{int }(S+C)$. Since $(R^n\backslash\text{int }(S+C)) \supset (X\backslash\text{int }(X+C)) = X_e \neq \emptyset$ and $X\backslash\text{int }(S+C)$ is compact, a minimizer of f on $X\backslash\text{int }(S+C)$ exists and solves problem (P). Denote by $\inf(P)$ the optimal value in problem (P). Since the feasible set of problem (P) includes the feasible set of problem (MP), we have $\inf(P) \leqslant \inf(MP) < +\infty$.

The dual problem of problem (P) is formulated as

$$(D)\begin{cases}\text{maximize} & g^H(u),\\ \text{subject to} & u \in (S+C)^\circ.\end{cases}$$

Since $S+C \subset X+C$, the feasible set of problem (D) includes $(X+C)^\circ$. Therefore, problem (D) is a relaxed problem of (DP). We note that the feasible set $(S+C)^\circ$ is a polytope because $S+C$ is a convex polyhedral set and $0 \in \text{int }(S+C)$. Hence, problem (D) is a quasi-convex maximization over a polytope $(S+C)^\circ$. There exists an optimal solution of problem (D) among the vertices of $(S+C)^\circ$. Denote by $\sup(D)$ the optimal values of problem (D). Since problem (D) is the dual problem of problem (P) and a relaxed problem of problem (DP), we obtain $\sup(D) = -\inf(P) \geqslant -\inf(MP) = \sup(DP) > -\infty$ (Thach et al. (1996) and Konno et al. (1997), Chapter 4). This implies that the origin is not optimal to problem (D) since $g^H(0) = -\infty$. Consequently, we can choose an optimal solution of problem (D) from $V((S+C)^\circ)\backslash\{0\}$. Let $E(C)$ be a finite set of extreme directions of C satisfying $C = \{x \in R^n : x = \sum_{y\in E(C)} \lambda_y y,\ \lambda_y \geqslant 0\}$. Then,

$$\begin{aligned}(S+C)^\circ &= S^\circ \cap C^\circ\\ &= \{u \in R^n : \langle u, z\rangle \leqslant 1\ \forall z \in V(S),\ \langle u, y\rangle \leqslant 0\ \forall y \in E(C)\}.\end{aligned}$$

LEMMA 3.1. *For any* $v \in V((S+C)^\circ)\backslash\{0\}$, $v \notin \text{int } X^\circ$.

Proof. Suppose to the contrary that there exists $v \in V((S+C)^\circ)\backslash\{0\}$ satisfying $v \in \text{int } X^\circ$. Then, since v is a vertex of $(S+C)^\circ$ and $(S_k+C)^\circ = \{u \in R^n : \langle u, z\rangle \leqslant 1\ \forall z \in V(S),\ \langle u, y\rangle \leqslant 0\ \forall y \in E(C)\}$,

$$\begin{aligned}\exists a^1, \ldots, a^n \in V(S) \cup E(C)\ \text{ such that }\ &\dim\{a^1, \ldots, a^n\} = n\\ &\text{and } \langle v, a^i\rangle = b_i\ i = 1, \ldots, n,\end{aligned} \tag{3}$$

where for all $i \in \{1, \ldots, n\}$,

$$b_i = \begin{cases}1 & \text{if } a^i \in V(S)\\ 0 & \text{if } a^i \in E(C).\end{cases}$$

Note that $u' \notin \text{int } X^\circ$ if a point $u' \in R^n$ satisfies that $\langle u', z\rangle \geqslant 1$ for some $z \in V(S)$, because $X^\circ \subset S^\circ = \{u \in R^n : \langle u, z\rangle \leqslant 1 \text{ for all } z \in V(S)\}$. Hence, by the assumption of v, $\{a^1, \ldots, a^n\} \subset E(C)$. Then v is the origin of R^n because

$\cap_{i=1}^{n}\{u \in R^n : \langle u, a^i\rangle = 0\} = \{0\}$ from (3). This is a contradiction and hence, we have $v \notin \text{int } X^\circ$ for any $v \in V((S+C)^\circ)\backslash\{0\}$. □

For any $v \in V((S+C)^\circ)\backslash\{0\}$, let x^v be an optimal solution of the following convex minimization problem:

$$(SP(v)) \begin{cases} \text{minimize} & f(x) \\ \text{subject to} & x \in X \cap \{x \in R^n : \langle v, x\rangle \geqslant 1\}. \end{cases}$$

By Lemma 3.1, the feasible set of problem $(SP(v))$ is not empty. Then, we have

$$\begin{aligned} g^H(v) &= -\inf\{g(x) : \langle v, x\rangle \geqslant 1\} \\ &= -\inf\{f(x) : \langle v, x\rangle \geqslant 1,\ x \in X\} \\ &= -\inf(SP(v)) \\ &= -f(x^v), \end{aligned}$$

where $\inf(SP(v))$ is the optimal value in problem $(SP(v))$. Hence, $\hat{v} \in V((S+C)^\circ)\backslash\{0\}$ is an optimal solution of problem (D) if $f(x^{\hat{v}}) = \min\{f(x^v) : v \in V((S+C)^\circ)\backslash\{0\}\}$. Moreover, $x^{\hat{v}}$ is optimal to problem (P) (Thach et al. (1996) and Konno et al. (1997), Proposition 4.3).

3.2. AN INNER APPROXIMATION ALGORITHM

The discussion in the previous subsection suggests the following inner approximation algorithm for problem (MP).

ALGORITHM IAM-(MP)

Initialization. Generate a finite set V_1 such that $V_1 \subset X$ and that $0 \in \text{int}(\text{co } V_1)$. Let $S_1 = \text{co } V_1$. Compute the vertex set $V((S_1+C)^\circ)$. For convenience, let $V((S_0+C)^\circ) = \{0\}$. Set $k \leftarrow 1$ and go to Step 1.

Step 1. For every $v \in V((S_k+C)^\circ)\backslash V((S_{k-1}+C)^\circ)$ let x^v be an optimal solution of problem $(SP(v))$. Choose $v^k \in V((S_k+C)^\circ)\backslash\{0\}$ satisfying $f(x^{v^k}) = \min\{f(x^v) : v \in V((S_k+C)^\circ)\backslash\{0\}\}$. Let $x(k) = x^{v^k}$.

Step 2. Solve the following convex minimization problem:

$$\begin{cases} \text{minimize} & \phi(x; v^k) = \max\{p(x), h(x, v^k)\} \\ \text{subject to} & x \in R^n \end{cases} \tag{4}$$

where $h(x, v^k) = -\langle v^k, x\rangle + 1$. Let z^k and α_k denote an optimal solution of problem (4) and the optimal value, respectively. and Lemma 3.3. It will be proved later in Theorem 3.1 that the optimal value of problem (4) exists and that $z^k \in X$, respectively.

(a) If $\alpha_k = 0$, then stop; v^k solves problem (DP). Moreover, $x(k)$ solves problem (MP) and the optimal value of problem $(SP(v^k))$ is the optimal value of problem (MP).

(b) Otherwise, set $V_{k+1} = V_k \cup \{z^k\}$. Let $S_{k+1} = \text{co } V_{k+1}$. Compute the vertex set $V((S_{k+1} + C)^\circ)$. Set $k \leftarrow k + 1$ and return to Step 1.

Note that S_k, $k = 1, 2, \dots$, are polytopes. Since $0 \in \text{int } (\text{co } V_1) = \text{int } S_1$, $S_k + C$, $k = 1, 2, \dots$, satisfy that $0 \in \text{int } (S_k + C)$. Moreover, S_k, $k = 1, 2, \dots$, are contained in X if $S_1 \subset X$ and $\{z^k\} \subset X$. Hence, we have $S_k \subset X$. This implies that the following problems (P_k) and (D_k) are relaxed problems of (MP) and (DP), respectively.

$$(P_k) \begin{cases} \text{minimize} & g(x) \\ \text{subject to} & x \in R^n \setminus \text{int } (S_k + C), \end{cases}$$

$$(D_k) \begin{cases} \text{maximize} & g^H(u) \\ \text{subject to} & u \in (S_k + C)^\circ. \end{cases}$$

From the discussion in Subsection 3.1, $x(k)$ and v^k obtained in Step 1 of the algorithm solve problems (P_k) and (D_k), respectively.

In Step (2b) of the algorithm, $V((S_{k+1}+C)^\circ)$ can be obtained from $V((S_k+C)^\circ)$ because $(S_{k+1} + C)^\circ = (S_k + C)^\circ \cap \{u \in R^n : \langle u, z^k\rangle \leqslant 1\}$. Since for any k, $(S_k + C)^\circ \subset C^\circ = \text{cone}\{c^1, \dots, c^K\}$, the sets $(S_k + C)^\circ$, $k = 1, 2, \dots$, are contained in the linear space generated by the K vectors $c^1, \dots, c^K$. Hence, the computation of the vertices of $(S_k + C)^\circ$ can be carried out in a space of dimension K which is generally much smaller than n (Chen et al. (1991), Horst and Tuy (1996), Konno et al. (1997) and Tuy (1998)).

For any k, the following assertions are valid.

- $V(S_k) \subset V_k$.
- $(S_k + C)^\circ = \{u \in R^n : \langle u, z\rangle \leqslant 1 \; \forall z \in V_k, \; \langle u, y\rangle \leqslant 0 \; \forall y \in E(C)\}$.
- $S_k + C = \{x \in R^n : \langle v, x\rangle \leqslant 1, \; \forall v \in V((S_k + C)^\circ)\}$.
- $(S_{k+1} + C)^\circ = (S_k + C)^\circ \cap \{u \in R^n : \langle u, z^k\rangle \leqslant 1\}$.

In the following subsections, we shall discuss the suitability of the algorithm:
Subsection 3.3: We propose a procedure generating an initial finite set V_1.
Subsection 3.4: We show the following: First, at every iteration of the algorithm, the minimal value of the objective function of problem (4) exists. Secondly, at every iteration of the algorithm, an optimal solution z^k of problem (4) is contained in X. Lastly, if the algorithm terminates after finitely many iterations, we obtain the optimal solutions of problems (MP) and (DP).

Subsection 3.5: We prove that every accumulation point of a sequence $\{v^k\}$ is an optimal solution to problem (DP) when the algorithm does not terminate after finitely many iterations. Moreover, every accumulation point of a sequence $\{x(k)\}$ is an optimal solution to problem (MP).

3.3. GENERATING AN INITIAL FINITE SET V_1

In this subsection, we propose a procedure for generating an initial finite set V_1 such that $V_1 \subset X$ and $0 \in \text{int}\,(\text{co}\, V_1)$.

Let

$$V = \{e^1, e^2, \ldots, e^n, e^{n+1}\}$$

where, for all $i \in \{1, \ldots, n\}$, $e^i \in R^n$ satisfies that $e^i_i = 1$ and $e^i_j = 0$ for all $j \neq i$, and $e^{n+1} \in R^n$ satisfies that $e^{n+1}_j = -1$ for all j. Then, $0 \in \text{int}\,(\text{co}\, V)$. But V is not always contained in X. To get an initial set contained in X, let for all $i \in \{1, \ldots, n+1\}$,

$$\lambda_i = \begin{cases} 1 & p(e^i) \leqslant 0, \\ \dfrac{-p(0)}{p(e^i) - p(0)} & p(e^i) > 0 \end{cases}$$

and let $\bar{V} := \{\lambda_1 e^1, \lambda_2 e^2, \ldots, \lambda_{n+1} e^{n+1}\}$. Then, $\lambda_i e^i \in X$ if $p(e^i) \leqslant 0$. Even if $p(e^i) > 0$, we can prove $\lambda_i e^i \in X$. Indeed, because p is a convex function, we have

$$p(\lambda_i e^i) = p(\lambda_i e^i + (1 - \lambda_i)0) \leqslant \lambda_i p(e^i) + (1 - \lambda_i) p(0) = 0.$$

Therefore $p(\lambda_i e^i) \leqslant 0$ for all $i \in \{1, \ldots, n+1\}$, that is, $\bar{V} \subset X$. Obviously, $0 \in \text{int}\,(\text{co}\, \bar{V})$. Consequently, we can set V_1 by $\bar{V}$.

3.4. STOPPING CRITERION OF ALGORITHM IAM-*(MP)*

In this subsection, we examine the suitability of the stopping criterion of Algorithm IAM-(MP).

THEOREM 3.1. *For every $v \in R^n$, the function $\phi(x; v)$ attains its minimum over R^n.*

Proof. Since p and $h(\,\cdot\,, v)$ are continuous, ϕ is continuous. We have $\{x \in R^n : \phi(x; v) \leqslant 1\} \neq \emptyset$ because $\phi(0; v) = 1$. By the definition of ϕ, $\{x \in R^n : \phi(x; v) \leqslant 1\} \subset \{x \in R^n : p(x) \leqslant 1\}$. Since p is a proper convex function and X is compact, $\{x \in R^n : p(x) \leqslant 1\}$ is compact (Rockafellar (1970), Corollary 8.7.1). This implies that $\{x \in R^n : \phi(x; v) \leqslant 1\}$ is compact. Consequently, the minimum value of ϕ over R^n exists (Hestenes (1975), Theorem 2.1). □

LEMMA 3.2. *At iteration k of Algorithm IAM-(MP), let $v^k \in V((S_k + C)^\circ)$ be an optimal solution for problem (D_k). Then $v^k \notin$ int X°.*

Proof. From the discussion in Subsection 3.1, $v^k \neq 0$, Moreover, from Lemma 3.1, so that $v^k \in V((S_k + C)^\circ)\backslash\{0\}$. we have $v^k \notin \text{int } X^\circ$. $\square$

LEMMA 3.3. *At iteration k of Algorithm IAM-(MP), assume that* $S_k \subset X$. *Then*

(i) $\alpha_k \leqslant 0$,
(ii) $z^k \in X$.

Proof. By Lemma 3.2, $v^k \notin \text{int } X^\circ$. Therefore, there is $\hat{x} \in X$ such that $\langle v^k, \hat{x} \rangle \geqslant 1$. Furthermore,

$$\alpha_k = \min_{x \in R^n} \phi(x; v^k) \leqslant \phi(\hat{x}; v^k) = \max\{p(\hat{x}), -\langle v^k, \hat{x} \rangle + 1\} \leqslant 0.$$

Since $p(z^k) \leqslant \alpha_k$, we obtain $p(z^k) \leqslant 0$. Consequently, $z^k \in X$. $\square$

From Lemma 3.3 and the definition of S_1 as in Subsection 3.3, we obtain the following inclusive relations:

- $S_1 + C \subset S_2 + C \subset \ldots \subset S_k + C \subset \ldots \subset X + C$,
- $(S_1 + C)^\circ \supset (S_2 + C)^\circ \supset \ldots \supset (S_k + C)^\circ \supset \ldots \supset (X + C)^\circ$.

Moreover, we note that sup $(D_{k-1}) \geqslant$ sup (D_k) for any $k \geqslant 2$, that is,

$$g^H(v^1) \geqslant g^H(v^2) \geqslant \cdots \geqslant g^H(v^k) \geqslant \cdots \geqslant \sup (DP), \tag{5}$$

and that inf $(P_{k-1}) \leqslant$ inf (P_k) for any $k \geqslant 2$, that is,

$$f(x(1)) \leqslant f(x(2)) \leqslant \cdots \leqslant f(x(k)) \leqslant \cdots \leqslant \inf (MP). \tag{6}$$

If the algorithm terminates at a Step (2a) then we obtain an optimal solution, as shown in the following:

THEOREM 3.2. *At iteration k of Algorithm IAM-*(MP), $\alpha_k = 0$ *if and only if* $v^k \in X^\circ$.

Proof. First, suppose that $\alpha_k = 0$. Then, $\max\{p(x), 1 - \langle v^k, x \rangle\} \geqslant 0$ for all $x \in R^n$. This implies that $\langle v^k, x \rangle \leqslant 1$ for all $x \in X$. Consequently, $v^k \in X^\circ$ if $\alpha_k = 0$.

Next, suppose that $v^k \in X^\circ$. Then, since $(X^\circ)^\circ = X$, we obtain $X \subset \{x \in R^n : \langle v^k, x \rangle \leqslant 1\}$. Therefore, $X \cap \{x \in R^n : \langle v^k, x \rangle > 1\} = \emptyset$, that is,

$$\nexists x \in R^n \text{ such that } p(x) < 0 \text{ and } -\langle v^k, x \rangle + 1 < 0.$$

Hence, for any $x \in R^n$, $\phi(x; v^k) \geqslant 0$, that is, $\alpha_k \geqslant 0$. Consequently, by Lemma 3.3 (i), $\alpha_k = 0$. $\square$

THEOREM 3.3. *At iteration k of Algorithm IAM-*(MP), *if* $\alpha_k = 0$, *then*

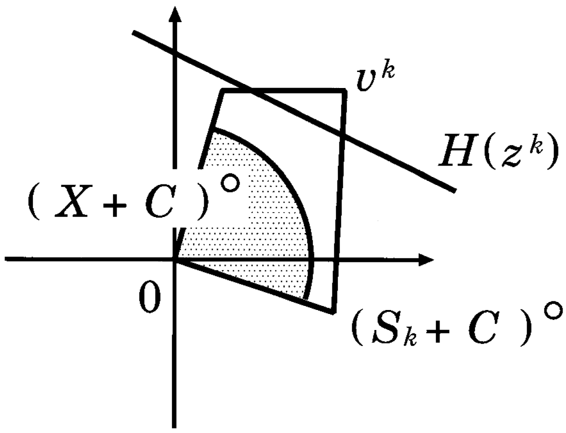

Figure 1. Generation of $S_{k+1}+C$ and $(S_{k+1}+C)^\circ$; $H(z^k)=\{u : \langle u, z^k\rangle = 1\}$.

(i) v^k *is an optimal solution of problem* (DP)*,*

(ii) $x(k)$ *is an optimal solution of problem* (MP)*.*

Proof. Suppose that $\alpha_k = 0$. Then, by Theorem 3.2, $v^k \in X^\circ$. We obtain $v^k \in X^\circ \cap C^\circ = (X+C)^\circ$ because $v^k \in (S_k+C)^\circ \subset C^\circ$. Therefore, $g^H(v^k) \leqslant \sup(DP)$. Since v^k is an optimal solution of (D_k) and $(S_k+C)^\circ \supset (X+C)^\circ$, we have $g^H(v^k) \geqslant \sup(DP)$. Hence, $g^H(v^k) = \sup(DP)$. Consequently, v^k is an optimal solution of problem (DP). Furthermore, $x(k)$ is an optimal solution of problem (MP) (Thach et al. (1996) and Konno et al. (1997), Proposition 4.3). □

At iteration k of Algorithm IAM-(MP), $\langle v^k, z^k\rangle > 1$ if $\alpha_k < 0$. Hence, $S_{k+1} + C = \text{co}\,(S_k \cup \{z^k\}) + C \neq S_k + C$ because $S_k + C \subset \{x \in R^n : \langle v^k, x\rangle \leqslant 1\}$. Moreover, since $V(S_{k+1}) \subset V(S_k) \cup \{z^k\}$, we have

$$(S_{k+1}+C)^\circ = (S_k+C)^\circ \cap \{u \in R^n : \langle u, z^k\rangle \leqslant 1\} \neq (S_k+C)^\circ \tag{7}$$

(see Figure 3.4).

REMARK 3.1. At iteration k of Algorithm IAM-(MP), for any $v \in V((S_{k+1} + C)^\circ)\backslash V((S_k+C)^\circ)$, $\langle v, z^k\rangle = 1$.

3.5. CONVERGENCE OF ALGORITHM IAM-(MP)

Algorithm IAM-(MP) using the stopping criterion discussed in Subsection 3.4 does not necessarily terminate after finitely many iterations. In this subsection, we consider the case where an infinite sequence $\{v^k\}$ is generated by the algorithm.

LEMMA 3.4. *Assume that $\{v^k\}$ is an infinite sequence such that for all k, v^k is an optimal solution of (D_k) at iteration k of Algorithm IAM-(MP). Then, there exists an accumulation point of $\{v^k\}$.*

Proof. Since $\{v^k\} \subset (S_1 + C)^\circ$ and $(S_1 + C)^\circ$ is compact, there exists an accumulation point of $\{v^k\}$. □

It follows from the following theorem that every accumulation point of $\{v^k\}$ belongs to the feasible set of problem (DP).

THEOREM 3.4. *Assume that $\{v^k\}$ is an infinite sequence such that for all k, v^k is an optimal solution of (D_k) at iteration k of Algorithm IAM-(MP) and that $\bar{v}$ is an accumulation point of $\{v^k\}$. Then $\bar{v}$ belongs to $(X + C)^\circ$.*

Proof. Let a subsequence $\{v^{k_q}\} \subset \{v^k\}$ converge to $\bar{v}$. Let z^{k_q} be an optimal solution of problem (4) at iteration k_q of the algorithm. Since $\{z^{k_q}\}$ belongs to the compact set X, it has an accumulation point $\bar{z}$. By taking a further subsequence if necessary, we may assume without loss of generality that $\{z^{k_q}\}$ converges to $\bar{z}$. Since $\{v^{k_q}\} \cap X^\circ = \emptyset$, by Theorem 3.2,

$$0 > \alpha_{k_q} = \max\{p(z^{k_q}), h(z^{k_q}, v^{k_q})\} \geqslant -\langle v^{k_q}, z^{k_q} \rangle + 1, \quad \text{for all } q.$$

Therefore, $\lim_{q\to\infty}\langle v^{k_q}, z^{k_q}\rangle = \langle \bar{v}, \bar{z}\rangle \geqslant 1$. On the other hand, since $v^{k_{q'}} \in (S_{k_{q+1}} + C)^\circ$ for all $q' > q$, and $(S_{k_{q+1}} + C)^\circ = (S_{k_q} + C)^\circ \cap \{u \in R^n : \langle u, z^{k_q}\rangle \leqslant 1\}$, we obtain $\lim_{q\to\infty}\langle v^{k_{q+1}}, z^{k_q}\rangle = \langle \bar{v}, \bar{z}\rangle \leqslant 1$. Hence,

$$\lim_{q\to\infty}\langle v^{k_q}, z^{k_q}\rangle = \langle \bar{v}, \bar{z}\rangle = 1, \text{ i.e., } \lim_{q\to\infty} h(z^{k_q}, v^{k_q}) = 0. \tag{8}$$

By Lemma 3.3, $\limsup_{q\to\infty}\alpha_{k_q} \leqslant 0$. Moreover, according to condition (8),

$$\liminf_{q\to\infty}\alpha_{k_q} = \liminf_{q\to\infty}\max\{p(z^{k_q}), h(z^{k_q}, v^{k_q})\} \geqslant \lim_{q\to\infty} h(z^{k_q}, v^{k_q}) = 0.$$

Consequently, $\lim_{q\to\infty}\alpha_{k_q} = 0$.

In order to obtain a contradiction, suppose that $\bar{v} \notin X^\circ$. Then, we have

$$\exists x' \in X \text{ such that } h(x', \bar{v}) = -\langle \bar{v}, x'\rangle + 1 < 0.$$

Since $h(\,\cdot\,, \bar{v})$ is a continuous function over R^n,

$$\exists \varepsilon > 0 \text{ such that } B(x', \varepsilon) \subset \{x \in R^n : h(x, \bar{v}) < 0\}$$

where $B(x', \varepsilon) = \{x \in R^n : \|x - x'\| < \varepsilon\}$. This implies that for any $\bar{x} \in (\text{int } X) \cap B(x', \varepsilon)$, $p(\bar{x}) < 0$ and $h(\bar{x}, \bar{v}) < 0$ because int $X \neq \emptyset$. Then, we obtain

$$\exists \delta > 0 \text{ such that } h(\bar{x}, v) < \frac{1}{2} h(\bar{x}, \bar{v}) < 0, \quad \forall v \in B(\bar{v}, \delta)$$

and, for any $v \in B(\bar{v}, \delta)$,

$$\begin{aligned} \min_{x \in R^n} \phi(x; v) &= \min_{x \in R^n} \max\{p(x), h(x, v)\} \\ &\leqslant \max\{p(\bar{x}), h(\bar{x}, v)\} \leqslant \max\left\{p(\bar{x}), \frac{1}{2} h(\bar{x}, \bar{v})\right\} < 0. \end{aligned}$$

Consequently, $\lim_{q\to\infty} \alpha_{k_q} \leqslant \max\{p(\bar{x}), h(\bar{x}, \bar{v})/2\} < 0$. This is a contradiction. Hence $\bar{v} \in X^\circ$. Moreover, since $\{v^{k_q}\} \subset (S_1 + C)^\circ \subset C^\circ$ and C° is a closed set, we have $\lim_{q\to\infty} v^{k_q} = \bar{v} \in C^\circ$. Therefore, we get that $\bar{v} \in (X + C)^\circ = (X^\circ) \cap (C^\circ)$. □

COROLLARY 3.1. *Assume that* $\{v^k\}$ *is an infinite sequence such that for all* k, v^k *is an optimal solution of problem* (D_k) *at iteration* k *of Algorithm IAM-*(MP) *and that* $\bar{v}$ *is an accumulation point of* $\{v^k\}$. *Then* $\bar{v} \notin$ int X°.

Proof. Let a subsequence $\{v^{k_q}\} \subset \{v^k\}$ converge to $\bar{v}$. By Lemma 3.2, $v^{k_q} \notin$ int X° for all q. Since $R^n \backslash \text{int } X^\circ$ is a closed set, we have $\lim_{q\to\infty} v^{k_q} = \bar{v} \in R^n \backslash \text{int } X^\circ$. □

Furthermore, the following theorem shows that every accumulation point of $\{v^k\}$ solves problem (DP).

THEOREM 3.5. *Assume that* $\{v^k\}$ *is an infinite sequence such that for all* k, v^k *is an optimal solution of problem* (D_k) *at iteration* k *of Algorithm IAM-*(MP) *and that* $\bar{v}$ *is an accumulation point of* $\{v^k\}$. *Then* $\bar{v}$ *solves problem* (DP). *Furthermore,* $\lim_{k\to\infty} g^H(v^k) = \sup(DP)$.

Proof. Let a subsequence $\{v^{k_q}\} \subset \{v^k\}$ converge to $\bar{v}$. Then v^{k_q}, $q = 1, 2, \ldots,$ and $\bar{v}$ are contained in C°. Since that f is continuous over R^n, h is continuous over $R^n \times R^n$, X is a compact set and $\{x \in R^n : \langle v, x\rangle \geqslant 1,\ x \in X\} = \{x \in R^n : -h(x, v) \geqslant 0,\ x \in X\} \neq \emptyset$ for any $v \in C^\circ \backslash (\text{int } X^\circ)$, we obtain that g^H is upper semicontinuous over $C^\circ \backslash (\text{int } X^\circ)$ (Hogan (1973)). Therefore, by (5),

$$g^H(\bar{v}) \geqslant \limsup_{q\to\infty} g^H(v^{k_q}) \geqslant \liminf_{q\to\infty} g^H(v^{k_q}) \geqslant \sup(DP).$$

By Theorem 3.4, $\bar{v} \in (X + C)^\circ$. Hence, $g^H(\bar{v}) \leqslant \sup(DP)$. Consequently,

$$g^H(\bar{v}) = \lim_{q\to\infty} g^H(v^{k_q}) = \sup(DP).$$

Furthermore, since $(S_1+C)^\circ$ is compact and includes $\{v^k\}$, we have $\lim_{k\to\infty} g^H(v^k) = \sup(DP)$. $\square$

In view of Theorems 3.4 and 3.5, every accumulation point of $\{v^k\}$ belongs to the feasible set of problem (DP) and solves problem (DP).

REMARK 3.2. At iteration k of Algorithm IAM-(MP), $X \cap \{x \in R^n : \langle v^k, x\rangle > 1\}$ is not empty if $\alpha_k < 0$. For any $x' \in X$ satisfying $\langle v^k, x'\rangle > 1$, there exists a $\lambda \in R$ such that $0 < \lambda < 1$ and $\lambda x' \in X \cap \{x \in R^n : \langle v^k, x\rangle \geqslant 1\}$. Then, from assumptions (B1) and (B2), we have

$$\begin{aligned} f(\lambda x') &= f(\lambda x' + (1-\lambda)0) \\ &\leqslant \lambda f(x') + (1-\lambda) f(0) < \lambda f(x') + (1-\lambda) f(x') = f(x'). \end{aligned}$$

Therefore, every optimal solution of problem $(SP(v^k))$ belongs to $\{x \in R^n : \langle v^k, x\rangle = 1\}$.

REMARK 3.3. For the feasible set $R^n \backslash \mathrm{int}(X+C)$ of (MP), we have

$$R^n \backslash \mathrm{int}(X+C) \supset X \backslash \mathrm{int}(X+C) \neq \emptyset.$$

THEOREM 3.6. *Assume that* $\{x(k)\}$ *is an infinite sequence such that for all* k, $x(k)$ *is an optimal solution of problem* (P_k) *at iteration* k *of Algorithm IAM-*(MP) *and that* $\bar{x}$ *is an accumulation point of* $\{x(k)\}$. *Then* $\bar{x}$ *belongs to* $R^n \backslash \mathrm{int}\,(X+C)$ *and solves problem* (MP). *Furthermore,* $\lim_{k\to\infty} g(x(k)) = \inf(MP)$.

Proof. Let a subsequence $\{x(k_q)\} \subset \{x(k)\}$ converge to $\bar{x}$. Then, there is a sequence $\{v^{k_q}\}$ such that v^{k_q} is an optimal solution of (D_{k_q}) at iteration k_q of the algorithm. By Remark 3.2, $\langle v^{k_q}, x(k_q)\rangle = 1$ for all q. Therefore, $\lim_{q\to\infty}\langle v^{k_q}, x(k_q)\rangle = \lim_{q\to\infty}\langle v^{k_q}, \bar{x}\rangle = 1$. Moreover, for every accumulation point $\bar{v}$ of $\{v^{k_q}\}$, $\langle \bar{v}, \bar{x}\rangle = 1$. By Theorem 3.4, since $\bar{v} \in (X+C)^\circ$, we obtain $\bar{x} \in \mathrm{bd}\,(X+C)$. Consequently, $\bar{x} \notin \mathrm{int}\,(X+C)$.

Since $x(k_q)$ is an optimal solution of problem $(SP(v^{k_q}))$, we get $g^H(v^{k_q}) = -g(x(k_q)) = -f(x(k_q))$ for all q. Therefore, by Theorem 3.5 and the continuity of f,

$$\begin{aligned} \inf(MP) &= -\sup(DP) = -\lim_{q\to\infty} g^H(v^{k_q}) = \lim_{q\to\infty} g(x(k_q)) \\ &= \lim_{q\to\infty} f(x(k_q)) = f(\bar{x}). \end{aligned}$$

Furthermore, since X is compact and includes $\{x(k)\}$, we have $\lim_{k\to\infty} g(x(k)) = \inf(MP)$. The proof is complete. $\square$

4. A Criterion for Finite Termination

In this section we investigate under which conditions Algorithm IAM-(MP) terminates after finitely many iterations.

Since every objective function of problem (MOP) is an affine function, the following theorem holds.

THEOREM 4.1. *(See Sawaragi et al. (1985), Theorems 3.5.3 and 3.5.4) Assume that problem* (MOP) *satisfies the Kuhn-Tucker constraint qualification at* $\bar{x} \in X$. *Then a necessary and sufficient condition for* $\bar{x}$ *to be a weakly efficient solution to problem* (MOP) *is that there exist* $\mu \in R^k$ *and* $\lambda \in R^t$ *such that*

(i) $-\sum_{i=1}^{K} \mu_i c^i + \sum_{j=1}^{t} \lambda_j \nabla p_j(\bar{x}) = 0,$

(ii) $\sum_{j=1}^{t} \lambda_j p_j(\bar{x}) = 0,$

(iii) $\mu \geqslant 0$, $\lambda \geqslant 0$ *and* $\mu_{i'} > 0$ *for some* $i' \in \{1, \ldots, K\}$.

REMARK 4.1. For any $\bar{x} \in X$ such that $p(\bar{x}) > p(0)$, problem (MOP) satisfies the Kuhn-Tucker constraint qualification at $\bar{x}$ because problem (MOP) satisfies assumptions (A1) and (A2).

Let $J(x) := \{j : p_j(x) = p(x),\ j = 1, \ldots, t\}$. Then, for any $x \in X$, the subdifferential $\partial p(x)$ of p at x is given by

$$\partial p(x) = \{y \in R^n : y = \sum_{j \in J(x)} \lambda_j \nabla p_j(x),\ \sum_{j=1}^{t} \lambda_j = 1,\ \lambda_j \geqslant 0\ \forall j \in J(x)\}.$$

By the principle of duality, C° is formulated as

$$\begin{aligned} C^\circ &= \{u \in R^n : u = \sum_{i=1}^{K} \mu_i c^i,\ \mu_i \geqslant 0\ i = 1, \ldots, K\} \\ &= \{u \in R^n : \langle u, y \rangle \leqslant 0\ \forall y \in E(C)\}. \end{aligned}$$

THEOREM 4.2. *Assume that problem* (MOP) *satisfies the Kuhn-Tucker constraint qualification at* $\bar{x} \in X$. *Then a necessary and sufficient condition for* $\bar{x} \in X$ *to be a weakly efficient solution to problem* (MOP) *is that* $\bar{x}$ *satisfies the following conditions:*

(a) $p(\bar{x}) = 0,$

(b) $C^\circ \cap \partial p(\bar{x}) \neq \emptyset.$

Proof. We shall show that $\bar{x} \in X$ satisfies conditions (a) and (b) if and only if for $\bar{x}$, there exist $\mu \in R^k$ and $\lambda \in R^t$ satisfying conditions (i), (ii) and (iii) in Theorem 4.1.

Suppose that $\bar{x}$ satisfies conditions (a) and (b). By condition (a), $J(\bar{x}) = \{j : p_j(\bar{x}) = 0,\ j = 1, \ldots, t\}$. Moreover, by condition (b),

$$\begin{aligned}\exists\mu \geqslant 0,\ \lambda_j \geqslant 0,\ j \in J(\bar{x}) \text{ such that } &\textstyle\sum_{i=1}^{K} \mu_i c^i = \sum_{j\in J(\bar{x})} \lambda_j \nabla p_j(\bar{x}) \\ &\text{and } \textstyle\sum_{j\in J(\bar{x})} \lambda_j = 1.\end{aligned} \tag{9}$$

Putting $\lambda_j = 0$ for all $j \notin J(\bar{x})$, we have $\sum_{j=1}^{t} \lambda_j p_j(\bar{x}) = 0$ and condition (i) in Theorem 4.1 is satisfied. From assumption (A1), condition (a) and the convexity of p_j, $j \in J(\bar{x})$, we have $0 > p_j(0) \geqslant p_j(\bar{x}) + \langle \nabla p_j(\bar{x}), 0 - \bar{x}\rangle = -\langle \nabla p_j(\bar{x}), \bar{x}\rangle$. This implies that $\nabla p_j(x) \neq 0$ for all $j \in J(\bar{x})$, and that $\sum_{j\in J(\bar{x})} \lambda_j \nabla p_j(\bar{x}) \neq 0$ because $\langle \sum_{j\in J(\bar{x})} \lambda_j \nabla p_j(\bar{x}), \bar{x}\rangle = \sum_{j\in J(\bar{x})} \lambda_j \langle \nabla p_j(\bar{x}), \bar{x}\rangle > 0$. Therefore we obtain $\sum_{i=1}^{K} \mu_i c^i \neq 0$ from (9), and $\mu_{i'} > 0$ for some $i' \in \{1, \ldots, K\}$. That is, μ and λ satisfy condition (iii) in Theorem 4.1.

Suppose that for $\bar{x} \in X$, there exist $\bar{\mu} \in R^k$ and $\bar{\lambda} \in R^t$ satisfying conditions (i), (ii) and (iii) in Theorem 4.1. Since $\bar{x} \in X$, $p_j(\bar{x}) \leqslant 0$ for all $j \in \{1, \ldots, t\}$, and $p_j(\bar{x}) < 0$ for all $j \notin J(\bar{x})$. Since int $C \neq \emptyset$, C° is pointed. Hence, by conditions (ii) and (iii),

$$\bar{\lambda}_j = 0 \text{ for all } j \notin J(\bar{x}). \tag{10}$$

Hence, $\sum_{j=1}^{t} \bar{\lambda}_j \nabla p_j(\bar{x}) = \sum_{j\in J(\bar{x})} \bar{\lambda}_j \nabla p_j(\bar{x})$. By condition (iii), $\sum_{i=1}^{K} \bar{\mu}_i c^i \neq 0$. Moreover, by conditions (i) and (iii),

$$0 \neq \sum_{i=1}^{K} \bar{\mu}_i c^i = \sum_{j=1}^{t} \bar{\lambda}_j \nabla p_j(\bar{x}) = \sum_{j\in J(\bar{x})} \bar{\lambda}_j \nabla p_j(\bar{x}). \tag{11}$$

This implies that $\bar{\lambda}_j > 0$ for some $j \in J(\bar{x})$, that is,

$$\sum_{j\in J(\bar{x})} \bar{\lambda}_j > 0. \tag{12}$$

Let $\Lambda := \sum_{j\in J(\bar{x})} \bar{\lambda}_j$ and $\lambda'_j := \frac{\bar{\lambda}_j}{\Lambda}$ for all $j \in J(\bar{x})$. Then, since $\sum_{j\in J(\bar{x})} \lambda'_j = 1$ and $\lambda'_j \geqslant 0$ for all $j \in J(\bar{x})$, we have $\sum_{j\in J(\bar{x})} \lambda'_j \nabla p_j(\bar{x}) \in \partial p(\bar{x})$. We notice that C° is a cone. Therefore, by condition (11),

$$\sum_{j\in J(\bar{x})} \lambda'_j \nabla p_j(\bar{x}) = \frac{1}{\Lambda} \sum_{i=1}^{K} \bar{\mu}_i c^i \in C^\circ.$$

Consequently, $\bar{x}$ satisfies condition (b). We remember that $p(\bar{x}) \leqslant 0$ because $\bar{x} \in X$. Therefore, by conditions (ii), (iii) and (10), and the definition of $J(\bar{x})$,

$$0 = \sum_{j=1}^{t} \bar{\lambda}_j p_j(\bar{x}) = \sum_{j\in J(\bar{x})} \bar{\lambda}_j p_j(\bar{x}) = \sum_{j\in J(\bar{x})} \bar{\lambda}_j p(\bar{x}) \leqslant 0.$$

By (12), we obtain $p(\bar{x}) = 0$. Consequently, $\bar{x}$ satisfies condition (a). $\square$

Note that for any $x \in R^n$, $\partial p(x)$ is a compact convex set. Moreover we note that $0 \notin \partial p(x)$ for any $x \in R^n$ such that $p(x) > p(0)$ because x is not a minimum point of a convex function p over R^n if $p(x) > p(0)$. Note that $E(C)$ is a finite set (see Subsection 3.2). Let $E(C) = \{y^1, \dots, y^m\}$. For $x \in X$ such that $p(x) > p(0)$, we consider the following problem:

$$(LP(x)) \begin{cases} \text{minimize} & \eta \\ \text{subject to} & \langle \sum_{j \in J(x)} \lambda_j \nabla p_j(x), y^i \rangle \leqslant \eta,\ i = 1, \dots, m, \\ & \sum_{j \in J(x)} \lambda_j = 1,\ \lambda_j \geqslant 0\ \forall j \in J(x). \end{cases}$$

Let $\eta(x)$ be the optimal value of problem $(LP(x))$. Then the following theorem holds.

THEOREM 4.3. *Assume that $x \in X$ satisfies that $p(x) > p(0)$ and $\eta(x) \leqslant 0$. Then $C^\circ \cap \partial p(x) \neq \emptyset$.*

Proof. Let $\bar{\lambda}_j$, $j \in J(x)$, optimize problem $(LP(x))$ and let $\bar{u} = \sum_{j \in J(x)} \bar{\lambda}_j \nabla p_j(x)$. It is obvious that $\bar{u}$ belongs to $\partial p(x)$. Since $\max_{y \in E(C)} \langle \bar{u}, y \rangle = \eta(x) \leqslant 0$, $\bar{u} \in C^\circ = \{u \in R^n : \langle u, y \rangle \leqslant 0\ \forall y \in E(C)\}$. Consequently $\bar{u} \in C^\circ \cap \partial p(x)$. $\square$

From Theorems 4.2 and 4.3, $x' \in X$ is a weakly efficient solution to problem (MOP) if and only if x' satisfies that $p(x') = 0$ and $\eta(x') \leqslant 0$. Since the feasible set of problem $(SP(v^k))$ is included in X, the sequence $\{x(k)\}$ generated by Algorithm IAM-(MP) belongs to X. Therefore, we get $p(x(k)) \leqslant 0$ for all k. Moreover, from Theorem 3.6, every accumulation point of $\{x(k)\}$ is contained in the feasible set of problem (MP). That is, every accumulation point of $\{x(k)\}$ is a weakly efficient solution to problem (MOP). Hence, we have that

$$p(\bar{x}) = 0 \text{ and } \eta(\bar{x}) \leqslant 0 \tag{13}$$

where $\bar{x}$ is an arbitrary accumulation point of $\{x(k)\}$.

Since p is continuous, from Theorem 4.2 we get that $\lim_{k\to\infty} p(x(k)) = 0$. Hence, since $p(0) < 0$, there exists k' such that $p(x(k')) > p(0)$. Let $\{\tau_k\}$ be a sequence of positive numbers such that $\lim_{k\to\infty} \tau_k = 0$, $\sum_{k=1}^{\infty} \tau_k = \infty$. Then, we get that $\lim_{k\to\infty} \sup \tau_k \eta_k \leqslant 0$. Therefore, it follows that for any $\gamma_1, \gamma_2 > 0$,

$$\exists k' \text{ such that } p(x(k')) > \max\{p(0), -\gamma_1\} \text{ and } \tau_{k'} \eta(x_{k'}) \leqslant \gamma_2. \tag{14}$$

According to (14), within tolerances $\gamma_1, \gamma_2 > 0$, the algorithm terminates after finitely many iterations. By doing this, the weak efficiency of the obtained solution is compromised, i.e., the weak efficiency condition $p(x) = 0$ and $\eta(x) \leqslant 0$ is relaxed to $p(x) > \max\{p(0), -\gamma_1\}$ and $\tau_k \eta(x) \leqslant \gamma_2$. From this point of view, we replace Step 2 of the algorithm by the following:

Step 2. Solve problem $LP(x(k))$. Let $\eta(x(k))$ be the optimal value of the problem.

(a) If $p(x(k)) > \max\{p(0), -\gamma_1\}$ and $\tau_k \eta(x(k)) \leqslant \gamma_2$, then stop; v^k and $x(k)$ are compromise solutions for problems (DP) and (MP), respectively.

(b) Otherwise, solve problem (4) for v^k. Let z^k be an optimal solution for the problem. Set $V_{k+1} = V_k \cup \{z^k\}$ and compute the vertex set $V((S_{k+1} + C)^\circ)$. Set $k \leftarrow k + 1$ and go to Step 1.

5. Identifying Redundant Constraints

To execute Algorithm IAM-(MP) proposed in Section 3, it is necessary to compute the vertex set of the feasible set $(S_k + C)^\circ$ of (D_k) in each step. Note that at each iteration a new point z^k is added to V_k, but no point is ever deleted. This means that the number of constraints for $(S_k + C)^\circ$ $(k = 1, 2, \ldots)$ increases from iteration to iteration. In this section, we propose a procedure for eliminating a redundant point for $(S_{k+1} + C)^\circ$ from V_k at iteration k.

5.1. AT INITIALIZATION IN ALGORITHM IAM-(MP)

Let V' be a finite set generated by the procedure suggested in Subsection 3.3 and let $S' = \operatorname{co} V'$. Then $(S')^\circ$ is formulated as $(S')^\circ = \{u \in R^n : \langle u, z\rangle \leqslant 1,\ \forall z \in V'\}$. A point $\bar{z} \in V'$ is regarded as redundant for $(S' + C)^\circ$ if $\bar{z}$ satisfies the following condition:

$$(S' + C)^\circ = T(\bar{z}), \tag{15}$$

where $T(\bar{z}) = C^\circ \cap \{u \in R^n : \langle u, z\rangle \leqslant 1,\ \forall z \in V' \backslash \{\bar{z}\}\}$. A necessary and and sufficient condition for $\bar{z} \in V'$ to satisfy condition (15) is $T(\bar{z}) \subset \{u \in R^n : \langle u, \bar{z}\rangle \leqslant 1\}$. These redundant points for $(S' + C)^\circ$ can be eliminated by the following theorem.

Let

$$H(z) := \{u \in R^n : \langle u, z\rangle = 1\} \quad \text{for all } z \in V',$$
$$H(y) := \{u \in R^n : \langle u, y\rangle = 0\} \quad \text{for all } y \in E(C).$$

THEOREM 5.1. *A point $\bar{z} \in V'$ satisfies condition (15) if and only if*

$$\exists z' \in V' \backslash \{\bar{z}\} \text{ such that } V((S' + C)^\circ \cap H(\bar{z})) \subset V((S' + C)^\circ \cap H(z')). \tag{16}$$

Proof. First, to prove the only if part, let $\bar{z} \in V'$ satisfy condition (15). If $(S' + C)^\circ \subset \{u \in R^n : \langle u, \bar{z}\rangle < 1\}$, $\bar{z}$ satisfies condition (16) because $(S' +$

$C)^\circ \cap H(\bar{z}) = \emptyset$. Otherwise, $(S' + C)^\circ \cap H(\bar{z}) \neq \emptyset$. Then, by condition (15), $(S' + C)^\circ \cap H(\bar{z}) \subset \text{bd}\,(S' + C)^\circ \subset \text{bd}\,T(\bar{z})$. Therefore, there exists $z' \in (V' \cup E(C))\backslash\{\bar{z}\}$ such that $(S' + C)^\circ \cap H(\bar{z}) \subset (S' + C)^\circ \cap H(z')$. Moreover, since $0 \notin (S'+C)^\circ \cap H(\bar{z})$ and $0 \in (S'+C)^\circ \cap H(y)$ for all $y \in E(C)$, we can suppose that $z' \in V'\backslash\{\bar{z}\}$. Consequently, since $V((S' + C)^\circ \cap H(\bar{z})) \subset V((S' + C)^\circ)$ and $V((S' + C)^\circ \cap H(z')) \subset V((S' + C)^\circ)$, we have $V((S' + C)^\circ \cap H(\bar{z})) \subset V((S' + C)^\circ \cap H(z'))$.

Next, to prove the if part, let $\bar{z} \in V'$ satisfy $V((S' + C)^\circ \cap H(\bar{z})) \subset V((S' + C)^\circ \cap H(z'))$ for some $z' \in V'\backslash\{\bar{z}\}$. Then, we have

$$(S' + C)^\circ \cap H(\bar{z}) \subset (S' + C)^\circ \cap H(z'). \tag{17}$$

In order to obtain a contradiction, suppose that $T(\bar{z}) \not\subset \{x \in R^n : \langle \bar{z}, x\rangle \leqslant 1\}$, that is,

$$\exists u' \in T(\bar{z}) \text{ such that } \langle u', \bar{z}\rangle > 1.$$

Since $u' \in T(\bar{z})$ and $z' \in V'\backslash\{\bar{z}\}$, we have $\langle u', z', \rangle \leqslant 1$. Let $\bar{u} := \lambda u' + (1-\lambda)0 = \lambda u'$ where $\lambda = 1/\langle u', \bar{z}\rangle$. Then $\bar{u} \in T(\bar{z})$ because $0 < \lambda < 1$ and $0 \in T(\bar{z})$. Since $\langle \bar{u}, \bar{z}, \rangle = 1$, we get $\bar{u} \in (S'+C)^\circ$. Therefore, $\bar{u} \in (S'+C)^\circ \cap H(\bar{z})$. However, since $\langle 0, z'\rangle < 1$ and $\langle u', z'\rangle \leqslant 1$, we obtain $\langle \bar{u}.z'\rangle < 1$, that is, $\bar{u} \notin (S' + C)^\circ \cap H(z')$. This contradicts condition (17). This completes the proof. $\square$

EXAMPLE 5.1. Consider the multiobjective programming problem:

$$\begin{cases} \text{maximize } \ \langle c^i, x\rangle, \ i = 1, 2 \\ \text{subject to } \ x \in X = \{x \in R^3 : p(x) \leqslant 0\} \end{cases}$$

where $c^1 = (1, 1, 2)^t$, $c^2 = (1, 1, -1)^t$ and $p(x) = x_1^2 + x_2^2 + x_3^2 - 1$.

By using the procedure proposed in Subsection 3.3, we obtain that

$$\lambda_1 e^1 = (1, 0, 0)^t, \ \lambda_2 e^2 = (0, 1, 0)^t, \ \lambda_3 e^3 = (0, 0, 1)^t,$$

$$\lambda_4 e^4 = \left(-\frac{2}{3}, -\frac{2}{3}, -\frac{2}{3}\right)^t$$

(obviously $V' = \{\lambda_i e^i : i = 1, 2, 3, 4\} \subset X$) and that $V((S' + C)^\circ) = \{v^1, v^2, v^3, v^4\}$ where

$$v^1 = (0, 0, 0)^t, \quad v^2 = \left(\frac{1}{2}, \frac{1}{2}, 1\right)^t, \quad v^3 = (1, 1, -1)^t, \quad v^4 = (1, 1, 1)^t.$$

Then,

$$\begin{aligned} &V((S' + C)^\circ \cap H(\lambda_1 e^1)) = \{v^3, v^4\}, \ \ V((S' + C)^\circ \cap H(\lambda_2 e^2)) = \{v^3, v^4\}, \\ &V((S' + C)^\circ \cap H(\lambda_3 e^3)) = \{v^2, v^4\}, \ \ V((S' + C)^\circ \cap H(\lambda_4 e^4)) = \emptyset. \end{aligned}$$

From Theorem 5.1, we can eliminate $\lambda_1 e^1$ (or $\lambda_2 e^2$) and $\lambda_4 e^4$ from V'. Hence, we set $V_1 = \{\lambda_2 e^2, \lambda_3 e^3\}$. Then, we obtain $(S_1 + C)^\circ = (S' + C)^\circ$. Notice that $0 \in \text{int}\,(S_1 + C)$.

5.2. AT ITERATION k OF ALGORITHM IAM-(MP)

At iteration k of Algorithm IAM-(MP), assume that for any $\bar{z} \in V_k$, $\bar{z}$ is non-redundant for $(S_k + C)^\circ$. Then, by the following theorem, we can generate a finite set V_{k+1} such that for every $z \in V_{k+1}$, the constraint $\langle u, z\rangle \leqslant 1$ is non-redundant for $(S_{k+1} + C)^\circ$.

Note that a point $\bar{z} \in V_k \cup \{z^k\}$ is non-redundant for $(S_{k+1} + C)^\circ$ if and only if $\bar{z}$ satisfies that

$$(S_{k+1} + C)^\circ \neq T_{k+1}(\bar{z}), \tag{18}$$

where $T_{k+1}(\bar{z}) = C^\circ \cap \{u \in R^n : \langle u, z\rangle \leqslant 1,\ \forall z \in (V_k \cup \{z^k\})\backslash\{\bar{z}\}\}$. Let

$$H_{\geqslant}(z^k) := \{u \in R^n : \langle u, z^k\rangle \geqslant 1\}.$$

LEMMA 5.1. *At iteration k of Algorithm IAM-(MP), z^k is non-redundant for $(S_{k+1} + C)^\circ$.*

Proof. Since $\langle v^k, z^k\rangle > 1$ and $\langle v^k, z\rangle \leqslant 1$ for all $z \in V_k$, it is obvious. □

LEMMA 5.2. At iteration k of Algorithm IAM-(MP), $\langle z^k, v\rangle = 1$ for all $v \in V((S_{k+1} + C)^\circ)\backslash V((S_k + C)^\circ)$.

Proof. Since $(S_{k+1} + C)^\circ = (S_k + C)^\circ \cap \{u \in R^n : \langle u, z^k\rangle \leqslant 1\}$, it is obvious. □

THEOREM 5.2. At iteration k of Algorithm IAM-(MP), assume that for any $z' \in V_k$, z' is non-redundant for $(S_k + C)^\circ$, that is,

$$(S_k + C)^\circ \neq C^\circ \cap \{u \in R^n : \langle u, z\rangle \leqslant 1,\ \forall z \in V_k\backslash\{z'\}\}. \tag{19}$$

Then, $\bar{z} \in V_k$ satisfies condition (18), i.e., $\bar{z}$ is non-redundant for $(S_{k+1} + C)^\circ$ if and only if

$$(V((S_k + C)^\circ)\backslash\{v^k\}) \cap H(\bar{z}) \not\subset (V((S_k + C)^\circ)\backslash\{v^k\}) \cap H_{\geqslant}(z^k). \tag{20}$$

Proof. First, to prove the only if part, let $\bar{z} \in V_k$ satisfy condition (18). Then, by Theorem 5.1, for all $z \in (V_k \cup \{z^k\})\backslash\{\bar{z}\}$,

$$V((S_{k+1} + C)^\circ \cap H(\bar{z})) \not\subset V((S_{k+1} + C)^\circ \cap H(z)).$$

Therefore, we have

$$\exists v' \in V((S_{k+1} + C)^\circ \cap H(\bar{z})) \text{ such that } \langle v', z^k\rangle < 1.$$

By Remark 3.1, v' belongs to $V((S_k+C)^\circ)$. Consequently, we get that $v' \in V((S_k+C)^\circ) \cap H(\bar{z})$ and $v' \notin V((S_k+C)^\circ) \cap H_{\geqslant}(z^k)$.

Next, to prove the if part, suppose that there is $\hat{v} \in V((S_k+C)^\circ) \cap H(\bar{z})$ satisfying $\langle \hat{v}, z^k \rangle < 1$. Then $\hat{v}$ is a vertex of $(S_{k+1}+C)^\circ$. Since $\langle \hat{v}, z^k \rangle < 1$,

$$\exists \varepsilon > 0 \text{ such that } \langle u, z^k \rangle < 1 \ \forall u \in B(\hat{v}, \varepsilon). \tag{21}$$

By condition (19),

$$\exists u' \in C^\circ \cap \{u \in R^n : \langle u, z \rangle \leqslant 1,\ \forall z \in V_k \backslash \{\bar{z}\}\} \text{ such that } \langle u', \bar{z} \rangle > 1.$$

Let $]\hat{v}, u'] := \{u \in R^n : u = \lambda \hat{v} + (1-\lambda)u',\ 0 \leqslant \lambda < 1\}$. Then $]\hat{v}, u'] \subset C^\circ \cap \{u \in R^n : \langle u, z \rangle \leqslant 1,\ \forall z \in V_k \backslash \{\bar{z}\}\}$. Moreover, since $\langle u', \bar{z} \rangle > 1$ and $\langle \hat{v}, \bar{z} \rangle = 1$, we obtain $]\hat{v}, u'] \subset \{u \in R^n : \langle u, \bar{z} \rangle > 1\}$. Hence, by condition (21), for any $\hat{u} \in]\hat{v}, u'] \cap B(\hat{v}, \varepsilon)$, we get that $\langle \hat{u}, \bar{z} \rangle > 1$, $\langle \hat{u}, z^k \rangle < 1$ and that $\hat{u} \in C^\circ \cap \{u \in R^n : \langle u, z \rangle \leqslant 1,\ \forall z \in V_k \backslash \{\bar{z}\}\}$. Consequently, since $]\hat{v}, u'] \cap B(\hat{v}, \varepsilon) \not\subset (S_{k+1}+C)^\circ$ and $]\hat{v}, u'] \cap B(\hat{v}, \varepsilon) \subset T_{k+1}(\bar{z})$, we get that $\bar{z}$ satisfies condition (18). $\square$

From Theorem 5.2, in the case of $(V((S_k+C)^\circ) \backslash \{v^k\}) \cap H_{\geqslant}(z^k) = \emptyset$ (i.e., $(V((S_k+C)^\circ) \cap H_{\geqslant}(z^k) = \{v^k\})$, by setting $V_{k+1} = V_k \cup \{z^k\}$, we obtain V_{k+1} such that every element is non-redundant for $(S_{k+1}+C)^\circ$. In the other cases, it is necessary to search out all points satisfying condition (20) from V_k.

6. Conclusion

In this paper, instead of solving problem (OES) directly, we have presented an inner approximation method. With a given tolerance for the weak efficiency to problem (MOP), the algorithm terminates after finitely many iterations.

To execute the algorithm, a convex minimization problem (4) is solved at each iteration. However, we note that it is not necessary to obtain an optimal solution for problem (4) at each step. At iteration k of the algorithm, it suffices to get a point which is contained in X and is not contained in $S_k + C$. That is, at each step, we can compromise solving problem (4) by getting a point z^k satisfying $\phi(z^k; v^k) < 0$, because z^k belongs to $X \backslash (S_k + C)$ if $\phi(z^k; v^k) < 0$.

By solving two kinds of convex minimization problems $(SP(v))$ and (4) successively, it is possible to obtain an approximate solution of problem (OES). These convex minimization problems are fairly easy to solve and therefore the proposed algorithm is practically useful.

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